A NON-STANDARD PROOF OF THE HAHN-BANACH EXTENSION THEOREM

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "A NON-STANDARD PROOF OF THE HAHN-BANACH EXTENSION THEOREM", submitted by S. K. JOLLY in partial fulfillment of the requirements of the degree of Master of Science.

ABSTRACT

The object of this thesis is to study the Hahn-Banach extension theorem of Functional Analysis by a method using non-standard analysis in the sense of A. Robinson [4]. In our first chapter we define a filter and state some theorems on filters, rings and fields which are required for the subsequent chapters. In some proofs given in this chapter, Zorn's lemma is used. In the second chapter a model for non-standard analysis is constructed in the form of ultrapowers, and we discuss infinitesimals and infinitely large numbers. By the instrumentality of this extended mode of analysis the Hahn-Banach extension theorem is proved in chapter III without any further use of Zorn's lemma.

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CHAPTER I

FILTERS, RINGS AND FIELDS

<u>Definition 1</u>. A non-empty set $\mathcal{F}$ of subsets of a non-empty set X is called a filter on X if it has the following properties:

(i) If $E \in \mathcal{F}$ and $X \supset F \supset E$ then $F \in \mathcal{F}$;

(ii) $E \in \mathcal{F}$, $F \in \mathcal{F}$ imply $E \cap F \in \mathcal{F}$;

(iii) $\emptyset \notin \mathcal{F}$.

From property (ii) it follows that a filter $\mathcal{F}$ is closed under finite intersection and so in view of (iii) $\mathcal{F}$ has the finite intersection property, that is, the intersection of any finite number of sets from $\mathcal{F}$ is non-empty. Since $\mathcal{F}$ is non-empty, (i) implies that $X \in \mathcal{F}$. Condition (iii) ensures that a filter cannot degenerate into the power set of X .

<u>Definition 2</u>. A filter $\mathcal{F}$ is called free if $\cap\mathcal{F} = \emptyset$. A filter that is not free is called fixed.

<u>Definition 3</u>. Let $\mathcal{F}$ and $\mathcal{F}'$ be two filters on X . If $\mathcal{F} \subset \mathcal{F}'$ then $\mathcal{F}'$ is called finer or larger than $\mathcal{F}$. If, in addition, $\mathcal{F} \neq \mathcal{F}'$ then $\mathcal{F}'$ is called strictly finer or larger than $\mathcal{F}$.

Observe that the collection of all the filters on X is partially ordered by set inclusion. We shall write $\mathcal{F} \leq \mathcal{F}'$ instead of $\mathcal{F} \subset \mathcal{F}'$.

<u>Definition 4</u>. A filter F on X is called an ultrafilter if there does not exist a filter on X which is strictly finer than F.

<u>Theorem 1</u>. Let F be a filter on X. Then there exists an ultrafilter U on X such that $F \leq U$.

<u>Proof</u>: Let S be the set of all filters on X finer than F. Then $S \neq \emptyset$, since $F \in S$. It is clear that S is partially ordered by set inclusion. Let T be any non-empty totally ordered subset of S. Then it is easily seen that $\cup T$ satisfies all the three axioms of Definition 1. Hence $\cup T$ is a filter on X. Obviously this filter is finer than F. Hence $\cup T \in S$. It is clear that for any $F_\alpha \in T$, $F_\alpha \leq \cup T$. Hence the filter $\cup T$ is an upper bound of T in S, and S is seen to be inductively ordered. By Zorn's lemma there must exist a maximal element in S. Let U be a maximal element in S. If U' is any filter on X which is finer than U, then $U' \geqq F$, hence $U' \in S$, and the maximality of U implies that $U' = U$. Hence U is a maximal element of the set of all filters on X or an ultrafilter on X such that $F \leq U$. This proves the theorem.

<u>Theorem 2</u>. Let U be an ultrafilter on X. If E and F are subsets of X such that $E \cup F \in U$, then $E \in U$ or $F \in U$.

<u>Proof</u>: Assume that $E \notin U$, $F \notin U$, and $E \cup F \in U$. Let $F = \{Y \mid Y \subset X, E \cup Y \in U\}$. Then

(i) If $Y_1 \in F$ then $E \cup Y_1 \in U$ and if $X \supset Y_2 \supset Y_1$ then

$E \cup Y_2 \supset E \cup Y_1$. Hence, by the first filter-property of $\mathcal{U}$, we get $E \cup Y_2 \in \mathcal{U}$. Hence $Y_2 \in \mathcal{F}$.

(ii) If $Y_1 \in \mathcal{F}$, $Y_2 \in \mathcal{F}$ then $E \cup Y_1 \in \mathcal{U}$, $E \cup Y_2 \in \mathcal{U}$. Hence, by the second filter-property of $\mathcal{U}$, $(E \cup Y_1) \cap (E \cup Y_2) \in \mathcal{U}$ or $E \cup (Y_1 \cap Y_2) \in \mathcal{U}$, which implies that $Y_1 \cap Y_2 \in \mathcal{F}$.

(iii) $E \cup \emptyset = E \notin \mathcal{U}$ (by assumption). This implies that $\emptyset \notin \mathcal{F}$. Hence $\mathcal{F}$ is a filter on X .

For any $A \in \mathcal{U}$, $X \supset E \cup A \supset A$, and so by the first filter-property of $\mathcal{U}$, we get $E \cup A \in \mathcal{U}$. This implies $A \in \mathcal{F}$. Hence $\mathcal{U} \leq \mathcal{F}$. Since $\mathcal{U}$ is an ultrafilter (maximal filter on X) there does not exist a filter on X which is strictly finer than $\mathcal{U}$. Therefore we must have $\mathcal{F} = \mathcal{U}$. Now, $E \cup F \in \mathcal{U}$ implies that $F \in \mathcal{F}$. Hence $F \in \mathcal{U}$. This contradicts the assumption that $F \notin \mathcal{U}$. Hence our assumption is wrong. This completes the proof of the theorem.

<u>Theorem 3</u>. Let $\mathcal{U}$ be an ultrafilter on X . If $\{E_i \mid 1 \leq i \leq n\}$ is a finite family of subsets of X such that $\cup\{E_i \mid 1 \leq i \leq n\} \in \mathcal{U}$ then there exists at least one member of this family which belongs to $\mathcal{U}$.

<u>Proof</u>: For $n = 2$, the result follows from Theorem 2. Suppose the theorem is true with n replaced by k , where $1 \leq k < n$. Let $\cup \{E_i \mid 1 \leq i \leq k+1\} \in \mathcal{U}$, then $\cup \{E_i \mid 1 \leq i \leq k\} \cup E_{k+1} \in \mathcal{U}$. By Theorem 2, either $E_{k+1} \in \mathcal{U}$ or $\cup \{E_i \mid 1 \leq i \leq k\} \in \mathcal{U}$. In the second case, by induction hypothesis, there is some E_i , $(1 \leq i \leq k)$ which belongs to $\mathcal{U}$. This shows that the theorem is true with n replaced

by $k+1$. Hence, by induction, the theorem is established.

Corollary. If $\cup\{E_i \mid 1 \leq i \leq n\} = X$, and $\mathcal{U}$ is any ultrafilter on X , then there exists an integer i such that $1 \leq i \leq n$ and $E_i \in \mathcal{U}$.

Theorem 4. $\mathcal{F}$ is an ultrafilter on X if and only if $\mathcal{F}$ is a filter on X , and $Y \in \mathcal{F}$ or $X-Y \in \mathcal{F}$ for every subset Y of X .

Proof: If $\mathcal{F}$ is an ultrafilter, then it is obviously a filter. Since $Y \cup (X-Y) = X$, the Corollary to Theorem 3, implies that $Y \in \mathcal{F}$ or $X-Y \in \mathcal{F}$. For the converse implication assume that $\mathcal{F}$ is not an ultra-filter. Then there exists a filter $\mathcal{F}'$ which is strictly finer than $\mathcal{F}$. Hence there exists a set $Y \in \mathcal{F}'$ such that $Y \notin \mathcal{F}$. By hypothesis this implies $X-Y \in \mathcal{F}$. Now, by the second filter-property of $\mathcal{F}'$, $E \cap Y \in \mathcal{F}'$ for all $E \in \mathcal{F}$. In particular, $(X-Y) \cap Y \in \mathcal{F}'$, that is, $\emptyset \in \mathcal{F}'$. This contradicts the third filter-property of $\mathcal{F}'$. Hence the assumption that $\mathcal{F}$ is not an ultrafilter is wrong. This proves that $\mathcal{F}$ is an ultrafilter on X .

Theorem 5. There exists a free ultrafilter on X if and only if X is infinite.

Proof: Let X be finite. Let $\mathcal{U}$ be an ultrafilter on X . Then $\mathcal{U}$ is finite. Hence $\cap\mathcal{U} \in \mathcal{U}$ by the second filter-property, and $\cap\mathcal{U} \neq \emptyset$ by the third filter-property. This shows that $\mathcal{U}$ is not free. Hence there exists no free ultrafilter on X . This proves that if there exists

a free ultrafilter on X , then X is infinite. Conversely, let X be infinite. Let $\mathcal{F} = \{E \mid E \subset X, X-E$ is finite$\}$. Then it is easily seen that $\mathcal{F}$ is a filter on X , and that $\cap\mathcal{F} = \emptyset$. By Theorem 1, there exists an ultrafilter $\mathcal{U}$ on X such that $\mathcal{F} \leq \mathcal{U}$. Consequently, $\cap\mathcal{U} \subset \cap\mathcal{F} = \emptyset$, which shows that $\mathcal{U}$ is a free ultrafilter on X .

<u>Theorem 6</u>. Let X be infinite. Let $\mathcal{U}$ be a free ultrafilter on X . Then every member of $\mathcal{U}$ is infinite.

<u>Proof</u>: Assume that a finite subset $\{x_1, x_2, \ldots, x_n\} = \cup_{i=1}^{n} \{x_i\}$ of X belongs to $\mathcal{U}$. Then, by Theorem 3, there exists at least one index k , $1 \leq k \leq n$, such that $\{x_k\} \in \mathcal{U}$. We now claim that $E \in \mathcal{U}$ implies that E contains $\{x_k\}$. To prove this claim, let there be an $E_o \in \mathcal{U}$ such that E_o does not contain $\{x_k\}$. Then, by the second filter-property $\{x_k\} \cap E_o \in \mathcal{U}$, or $\emptyset \in \mathcal{U}$, which contradicts the third filter-property. This establishes the above claim. Consequently, $\cap\mathcal{U} = \{x_k\} \neq \emptyset$. This contradicts the hypothesis that $\mathcal{U}$ is free. Hence our assumption is wrong, and the theorem is proved.

<u>Theorem 7</u>. Let Λ be any set and $\mathcal{F}$ be a non-empty set of subsets of Λ with the finite intersection property. Then $\mathcal{F}$ is included in some ultrafilter on Λ .

<u>Proof</u>: Let $\overline{\mathcal{F}}$ be the set of all subsets Δ of Λ with the property that there exists a non-void finite subset $\mathcal{G}$ of $\mathcal{F}$ such that $\Delta \supset \cap\mathcal{G}$. Then obviously $\mathcal{F} \subset \overline{\mathcal{F}}$. It is easily seen that $\overline{\mathcal{F}}$ has the

first filter-property. Let Δ_1 and Δ_2 be elements of $\overline{F}$. Let G_1 and G_2 be non-void finite subsets of F such that $\Delta_1 \supset \cap G_1$ and $\Delta_2 \supset \cap G_2$. Then $G_1 \cup G_2$ is a non-void finite subset of F such that $\Delta_1 \cap \Delta_2 \supset \cap(G_1 \cup G_2)$. This implies that $\Delta_1 \cap \Delta_2$ is an element of $\overline{F}$, showing that $\overline{F}$ has the second filter-property. Since F has the finite intersection property, from the definition of $\overline{F}$ it is clear that $\emptyset \notin \overline{F}$. Hence $\overline{F}$ has the third filter-property.

Consequently $\overline{F}$ is a filter on Λ containing F . By Theorem 1 there exists an ultrafilter U on Λ which includes $\overline{F}$, hence F .

In the following discussion R and R' are rings, and 0 and 0' are the additive identities of R and R' , respectively. We denote by 1 the multiplicative identity of R , if it exists, and by 1' the multiplicative identity of R' , if it exists. An element r of R will be called regular if it has a multiplicative inverse, and singular if it has no multiplicative inverse.

Definition 5. A non-empty subset I of R is said to be a (two-sided) ideal of R if:

(i) I is a subgroup of R under addition.

(ii) For every $x \in I$ and $r \in R$, both xr and rx are in I .

Remark. It is easy to see that (i) holds if and only if $x,y \in I$ implies that $x-y \in I$. I is called a proper ideal of R if $I \neq R$. Further note that if I is a proper ideal of R then no regular element

of R belongs to I .

<u>Definition 6</u>. A mapping h from the ring R into the ring R' is said to be a homomorphism if:

(i) $h(a+b) = h(a) + h(b)$ for all $a,b \in R$, and

(ii) $h(ab) = h(a)h(b)$ for all $a,b \in R$.

Note that the $+$ and $\cdot$ occurring on the left-hand sides of the relations in (i) and (ii) are those of R whereas the $+$ and $\cdot$ occurring on the right-hand sides are those of R' . It is easily seen that (i) implies that $h(0) = 0'$.

If h is a homomorphism of R into R' then by the kernel of h is meant the set of all elements a of R such that $h(a) = 0'$. A homomorphism of R into R' is called an isomorphism if it is a one-to-one mapping.

<u>Definition 7</u>. A proper ideal of R is said to be a maximal ideal of R if it is not properly contained in any other proper ideal of R .

In what follows hereafter, R is a commutative ring with multiplicative identity, 1 .

<u>Theorem 8</u>. If I is an ideal of R , then I is maximal if and only if R/I is a field.

This is a well-known theorem of algebra; therefore its proof will be omitted.

Theorem 9. Every ideal I of R is contained in some maximal ideal of R.

Proof: Let S be the set of all proper ideals of R which contain I. S is partially ordered with respect to set inclusion. To see that it is inductively ordered, let T be any totally ordered subset of S; then $\cup T \neq R$, because $1 \notin I_\alpha$ for any $I_\alpha \in T$; $\cup T$ is an ideal, and $\cup T \supset I$. Hence $\cup T$ is a proper ideal containing I, and therefore, belongs to S. Now it is obvious that $\cup T$ is an upper bound of T in S. By Zorn's lemma, S has a maximal element. Let M be a maximal element of S. Then $M \supset I$, and it is easily seen that M is a maximal ideal of R.

Theorem 10. If R has the property that the set of all non-regular elements of R is an ideal I, then I is maximal.

Proof: This theorem is obvious in view of the fact that a regular element does not belong to any proper ideal.

Definition 8. R is called an ordered ring (a totally ordered ring) if R is ordered (totally ordered) and satisfies the following two conditions:

(i) $x \leq y$ implies $x+z \leq y+z$,

(ii) $x \geq 0$, $y \geq 0$ imply $xy \geq 0$.

If R is a totally ordered ring and $a \in R$, let the sign $|a|$ denote $\max\{a,-a\}$; if a and b belong to R then, obviously,

$|a+b| \leq |a| + |b|$, and $|ab| = |a||b|$.

The following Theorems 11 to 14 are given in Luxemburg [3 (2)] in slightly different formulations and setting.

Theorem 11. If F is a totally ordered field, then F is archimedean if and only if F is ring and order isomorphic to a subfield of the field of real numbers.

Proof: If F is ring and order isomorphic to a subfield of the field of real numbers, F is archimedean because the field of real numbers and all its subfields are archimedean.

Conversely, suppose F is an archimedean totally ordered field. Let Q be the field of the rationals of F . Let R be the field of the Dedekind cuts of Q . Then we may suppose, without loss of generality, that the field of real numbers is identical with R . Let $x,y \in F$, and $x < y$; then $y-x > 0$ and $\frac{1}{y-x}$ exists. Since F is archimedean, there exists an integer n of F , such that $\frac{1}{y-x} < n$. Let m be the smallest integer k such that $k > nx$. Then $nx < m \leq nx+1$ and so $x < \frac{m}{n} \leq x + \frac{1}{n} < x + (y-x) = y$. Hence Q is dense in F . Hence every element x of F is uniquely determined by $\{r \mid r \in Q, r < x\}$. In other words, the mapping $\theta : F \to R$ defined by $x \to \theta(x)$, where $\theta(x)$ is the Dedekind cut whose lower class is $\{r \mid r \in Q, r < x\}$, is one-to-one. From the definitions of addition and multiplication of cuts it easily follows that $\theta(x+y) = \theta(x) + \theta(y)$ and $\theta(xy) = \theta(x)\theta(y)$. Hence θ is a ring isomorphism of F onto a subfield of R . It is

also obvious that θ is an order isomorphism of F onto this subfield. This completes the proof of the theorem.

It may be noted here, in passing, that the mapping θ considered in the above proof is the only mapping of F into R which is both a ring and an order isomorphism.

Let R be an ordered ring and let I be a proper ideal of R. Let h be the canonical mapping of R onto R/I, the residue class ring of R modulo I. Then $h(a) = I+a$ for every element a of R. In particular, $h(0) = I$. In the following, we shall sometimes write $0'$ instead of I. With this notation, $0'$ is the zero element of R/I, and $h(0) = 0'$.

<u>Definition 9</u>. We define a relation $\boxed{\leq}$ on R/I as follows:

If $h(a)$ and $h(b)$ be elements of R/I, then $h(a) \boxed{\leq} h(b)$ if and only if there exist elements x and y of $h(a)$ and $h(b)$, respectively, such that $x \leq y$.

In other words, $h(a) \boxed{\leq} h(b)$ if and only if there exist elements, x and y, of R such that $x \leq y$, $x-a \in I$ and $y-b \in I$.

<u>Remarks</u>. (i) We shall write $h(a) \leq h(b)$ instead of $h(a) \boxed{\leq} h(b)$. Of course, $h(a) < h(b)$ means that $h(a) \boxed{\leq} h(b)$ and $h(a) \neq h(b)$.

(ii) Since $a \in I+a = h(a)$, $b \in I+b = h(b)$, $a \leq b$ implies $h(a) \leq h(b)$.

(iii) It is easily seen that $h(c) \geq 0'$ if and only if there exists an element z of R such that $z \geq 0$ and $c-z \in I$.

(iv) $h(a) \leq h(b)$ if and only if $h(b-a) \geq 0'$.

Proof of (iv): If $h(a) \leq h(b)$ then by the above definition there exist elements $x,y \in R$ such that $x \leq y$ and $x-a \in I$, $y-b \in I$. This implies that $y-x \geq 0$ and $(b-a) - (y-x) \in I$. By (iii) we get $h(b-a) \geq 0'$. Conversely, if $h(b-a) \geq 0'$ then there exists an element z of R such that $z \geq 0$ and $(b-a) - z \in I$. Let $x = a$, $y = a+z$. Then $x \in h(a)$, $y-b = -((b-a)-z) \in I$, so that $y \in h(b)$, and clearly $x \leq y$. Hence $h(a) \leq h(b)$.

Theorem 12. The relation $\boxed{\leq}$ has the following properties:

(i) It is reflexive.

(ii) It is transitive.

(iii) It is antisymmetric if and only if I satisfies the following condition:

If $0 \leq x \leq y$ and $y \in I$, then $x \in I$.

(iv) $h(a) \leq h(b)$ implies $h(a) + h(c) \leq h(b) + h(c)$.

(v) $h(a) \geq 0'$ and $h(b) \geq 0'$ imply $h(a)h(b) \geq 0'$.

Proof: (i) This is obvious from remark (ii).

(ii) Let $h(a) \leq h(b)$ and $h(b) \leq h(c)$. Then by the definition of $\boxed{\leq}$, there exist elements $i_1, i_2, i_3, i_4 \in I$ such that $i_1+a \leq i_2+b$ and $i_3+b \leq i_4+c$, whence we get $(i_1-i_2)+a \leq b \leq (i_4-i_3)+c$. Hence $i+a \leq i'+c$, where i and $i' \in I$. This implies that $h(a) \leq h(c)$.

This proves transitivity of $\boxed{\leq}$.

(iii) Let us first assume that $\boxed{\leq}$ is antisymmetric. Let $0 \leq x \leq y$, and $y \in I$. Then $h(0) \leq h(x) \leq h(y)$ by remark (ii), and $h(y) = 0'$. Hence $0' \leq h(x) \leq 0'$. By the antisymmetry of $\boxed{\leq}$, $h(x) = 0'$. Hence $x \in I$. Conversely, suppose that $0 \leq x \leq y$ and $y \in I$ imply that $x \in I$. Let $h(a) \leq h(b)$ and $h(b) \leq h(a)$. Then by remark (iv) $h(b-a) \geq 0'$ and $h(a-b) \geq 0'$. By remark (iii) there exist elements, x and y , of R such that $x \geq 0$, $y \geq 0$, $(a-b)-x \in I$ and $(b-a)-y \in I$. From this we have $0 \leq x \leq x+y$ and $x+y \in I$. By our hypothesis, this implies that $x \in I$. Now, $(a-b)-x \in I$ and $x \in I$ imply that $a-b \in I$, which, in its turn, implies that $h(a-b) = 0'$. Hence $h(a) = h(b)$. Hence $\boxed{\leq}$ is antisymmetric.

(iv) Let $h(a) \leq h(b)$. Then, by remark (iv), $h(b-a) \geq 0'$, hence $h((b+c) - (a+c)) \geq 0'$, hence $h(a+c) \leq h(b+c)$, and so $h(a) + h(c) \leq h(b) + h(c)$.

(v) Let $h(a) \geq 0'$ and $h(b) \geq 0'$. Then there exist elements, x and y , of R such that $x \geq 0$, $y \geq 0$, $a-x \in I$ and $b-y \in I$, whence we get $xy \geq 0$, $ab-xb \in I$, $xb-xy \in I$, and so $ab-xy \in I$. This implies that $h(ab) \geq 0'$. Hence $h(a) \cdot h(b) \geq 0'$.

<u>Remark</u>. From the above theorem it is evident that $\boxed{\leq}$ is an ordering of R/I provided that the ideal I of R is such that $0 \leq x \leq y$ and $y \in I$ imply that $x \in I$.

<u>Theorem 13</u>. Let $0 \leq x \leq y$ and $y \in I$ imply that $x \in I$. Then R/I is an ordered ring under the ordering $\boxed{\leq}$, and the canonical

mapping h of R onto R/I is an order homomorphism.

Proof: This theorem is obvious from Theorem 12.

Theorem 14. Let R be a totally ordered ring, and let I be a proper ideal of R such that $0 \leq x \leq y$ and $y \in I$ imply that $x \in I$. Then R/I is a totally ordered ring. If R is archimedean, then R/I is also archimedean.

Proof: Theorem 14 follows easily from Theorem 13.

CHAPTER II

CONSTRUCTION OF A MODEL FOR NON-STANDARD ANALYSIS

Let R be the set of real numbers, let Λ be an arbitrary infinite set, and let R^Λ be the set of all mappings of Λ into R. Let us define addition and multiplication of elements of R^Λ componentwise. It is obvious that, with these operations of addition and multiplication, R^Λ is a ring whose zero element is $(\lambda \to 0 | \lambda \in \Lambda)$. It is also obvious that the unit element of this ring exists and equals $(\lambda \to 1 | \lambda \in \Lambda)$. Let A and B be elements of R^Λ. Then we say that $A \leq B$ if and only if $A(\lambda) \leq B(\lambda)$ for all $\lambda \in \Lambda$. This definition makes R^Λ an ordered ring.

In what follows, U is an ultrafilter on Λ.

<u>Definition 10</u>. Let A and B be two elements of R^Λ. We say that A is congruent to B relative to U, and write $A \equiv_U B$, if and only if $\{\lambda | \lambda \in \Lambda, A(\lambda) = B(\lambda)\} \in U$. We shall regard $\equiv_U$ as the sign of a relation on R^Λ.

<u>Theorem 15</u>. $\equiv_U$ is an equivalence relation on R^Λ.

<u>Proof</u>: (i) For any element A of R^Λ, $\{\lambda | \lambda \in \Lambda, A(\lambda) = A(\lambda)\} = \Lambda$, which belongs to U by a remark made after Definition 1. Hence $A \equiv_U A$. This proves that $\equiv_U$ is reflexive.

(ii) It is obvious that $\equiv_U$ is symmetric.

(iii) Let $A \equiv_U B$ and $B \equiv_U C$. Then $\{\lambda|\lambda \in \Lambda, A(\lambda) = B(\lambda)\} \in U$, and $\{\lambda|\lambda \in \Lambda, B(\lambda) = C(\lambda)\} \in U$. Now, clearly

$$\{\lambda|\lambda \in \Lambda, A(\lambda) = B(\lambda)\} \cap \{\lambda|\lambda \in \Lambda, B(\lambda) = C(\lambda)\} \subset \{\lambda|\lambda \in \Lambda, A(\lambda) = C(\lambda)\} .$$

Since the left hand side of this inclusion relation belongs to U by the second filter-property, the right hand side belongs to U by the first, showing that $A \equiv_U C$. Hence $\equiv_U$ is transitive. This completes the proof of the theorem.

<u>Theorem 16</u>. If A, B, A' and B' are elements of R^Λ such that $A \equiv_U A'$ and $B \equiv_U B'$, then $A+B \equiv_U A'+B'$ and $AB \equiv_U A'B'$.

<u>Proof</u>: $A \equiv_U A'$ implies that $\{\lambda|\lambda \in \Lambda, A(\lambda) = A'(\lambda)\} \in U$, and $B \equiv_U B'$ implies that $\{\lambda|\lambda \in \Lambda, B(\lambda) = B'(\lambda)\} \in U$. Now, clearly,

$$\{\lambda|\lambda \in \Lambda, A(\lambda) = A'(\lambda)\} \cap \{\lambda|\lambda \in \Lambda, B(\lambda) = B'(\lambda)\} \subset$$
$$\{\lambda|\lambda \in \Lambda, A(\lambda)+B(\lambda) = A'(\lambda)+B'(\lambda)\} .$$

By an argument used in the proof of Theorem 15, $A+B \equiv_U A'+B'$. Likewise, we get $AB \equiv_U A'B'$.

<u>Definition 11</u>. If A is an element of R^Λ, by $[A]$ is meant the equivalence class with respect to $\equiv_U$ determined by A. The set $R^\Lambda/\equiv_U$, of all equivalence classes with respect to $\equiv_U$ is denoted by S^*. In other words, $S^* = \{[A]|A \in R^\Lambda\}$.

Elements of S^* will be denoted by lower case Roman letters.

We now give a definition of addition and multiplication in S^* .

<u>Definition 12</u>. If $[A]$ and $[B]$ be any two elements of S^* , then (i) $[A] + [B] = [A+B]$ and (ii) $[A][B] = [AB]$.

If $[A] = [A']$ and $[B] = [B']$, then $A \equiv_{U} A'$ and $B \equiv_{U} B'$; by Theorem 16, $[A+B] = [A'+B']$ and $[AB] = [A'B']$. This shows that addition and multiplication of elements of S^* are well defined by (i) and (ii).

<u>Theorem 17</u>. With the operations of addition and multiplication defined by Definition 12, S^* is a commutative ring with a unit. The zero and the unit of this ring are $[(\lambda \to 0 | \lambda \in \Lambda)]$ and $[(\lambda \to 1 | \lambda \in \Lambda)]$, respectively. $(C \to [C] | C \in R^{\Lambda})$ is a ring homomorphism of R^{Λ} onto S^* . Let I be the kernel of this homomorphism. Then $I = [(\lambda \to 0 | \lambda \in \Lambda)]$, I an ideal of R^{Λ} , $S^* = R^{\Lambda}/I$, and $(C \to [C] | C \in R^{\Lambda})$ is the canonical mapping of R^{Λ} onto R^{Λ}/I .

<u>Proof</u>: This theorem is a special case of a well-known theorem of ring-theory.

<u>Corollary</u>. I is the set of all elements A of R^{Λ} such that $\{\lambda | \lambda \in \Lambda, A(\lambda) = 0\} \in U$.

It may be noted that, so far, only the hypothesis that U is a filter has been used, while the hypothesis that U is an ultrafilter

will be used but in the proof of the following Theorem 18.

Theorem 18. S^* is a field.

Proof: Let $0^* = [(\lambda \to 0 | \lambda \in \Lambda)]$ and $1^* = [(\lambda \to 1 | \lambda \in \Lambda)]$. Then 0^* and 1^* are the zero and unit elements of S^* respectively (and $0^* = I$) . Let a be any element of S^* different from 0^* . Let $a = [A]$, where $A \in R^{\Lambda}$. Then $\{\lambda | \lambda \in \Lambda, A(\lambda) = 0\} \notin U$. Since U is an ultrafilter, Theorem 4 implies that $\{\lambda | \lambda \in \Lambda, A(\lambda) \neq 0\} \in U$. Now, define

$$B(\lambda) = \begin{cases} \dfrac{1}{A(\lambda)} & \text{for } \lambda \in \Lambda, A(\lambda) \neq 0 , \\ 1 & \text{for } \lambda \in \Lambda, A(\lambda) = 0 . \end{cases}$$

and $b = [B]$.

Then, $\{\lambda | \lambda \in \Lambda, A(\lambda)B(\lambda) = 1\} = \{\lambda | \lambda \in \Lambda, A(\lambda) \neq 0\}$. Since this set belongs to U , it follows that

$$AB \equiv_U (\lambda \to 1 | \lambda \in \Lambda) .$$

Hence $[A][B] = 1^*$, and so $ab = 1^*$. Hence S^* is a field.

Definition 13. We define a relation $\boxed{\leq}$ on S^* by requiring that $[A] \boxed{\leq} [B]$ if and only if there exist elements, X and Y , of $[A]$ and $[B]$, respectively, such that $X \leq Y$.

We shall write $a \leq b$ instead of $a \boxed{\leq} b$. $a < b$ will mean that $a \boxed{\leq} b$ and $a \neq b$. It is easily seen that $[A] \leq [B]$ if and

only if $\{\lambda | \lambda \in \Lambda, A(\lambda) \leq B(\lambda)\} \in U$. Also, $[A] < [B]$ if and only if $\{\lambda | \lambda \in \Lambda, A(\lambda) < B(\lambda)\} \in U$. We note that the remarks on Definition 9 and Theorem 12 apply here.

Theorem 19. $\boxed{\leq}$ makes S^* a totally ordered field.

Proof: By Theorem 12, $\boxed{\leq}$ is reflexive and transitive. To prove its antisymmetry, let $(\lambda \to 0 | \lambda \in \Lambda) \leq X \leq Y$ and $Y \in I$, where I is the ideal of R^Λ defined in Theorem 17. Then $\{\lambda | \lambda \in \Lambda, Y(\lambda) = 0\} \in U$. Hence $\{\lambda | \lambda \in \Lambda, X(\lambda) = 0\} \in U$. This implies that $X \in I$. By Theorem 12 (iii), $\boxed{\leq}$ is antisymmetric. By Theorem 13, $\boxed{\leq}$ makes S^* an ordered field. Finally, to prove that $\boxed{\leq}$ is a total ordering of S^* , let $a \not\leq b$. Then for all $A \in a$ and all $B \in b$, $\{\lambda | \lambda \in \Lambda, A(\lambda) \leq B(\lambda)\} \notin U$. Since U is an ultrafilter Theorem 4 implies that $\{\lambda | \lambda \in \Lambda, A(\lambda) > B(\lambda)\} \in U$. Hence $\{\lambda | \lambda \in \Lambda, A(\lambda) \geq B(\lambda)\} \in U$. Therefore, $a \geq b$. The proof is now complete.

For any element r of R , let $r^* = [(\lambda \to r | \lambda \in \Lambda)]$. Then the mapping $(r \to r^* | r \in R)$ of R into S^* is obviously one-to-one and has the following properties:

(i) $(r_1+r_2)^* = r_1^* + r_2^*$,

(ii) $(r_1 r_2)^* = r_1^* r_2^*$,

(iii) $r_1^* \leq r_2^*$ if and only if $r_1 \leq r_2$.

The following theorem is now obvious.

<u>Theorem 20</u>. Let $\rho = (r \to r^* | r \in R)$ and $R^* = \{r^* | r \in R\}$. Then R^*, with the operations of addition and multiplication defined by Definition 12, is a subfield of S^*, and ρ is a ring and order isomorphism of R onto this subfield.

Hence the field R^* is ring and order isomorphic to the field R of real numbers.

Let N be the set of all positive integers of R and let $N^* = \{n^* | n \in N\}$. Then N^* is the set of the positive integers of S^*, hence of R^*.

The elements of S^* will be called numbers.

<u>Definition 14</u>. A number a is called infinitely large or infinite if there exists an element A of a such that A is unbounded on every element of $\mathcal{U}$. a is called infinitely small or infinitesimal if it is the reciprocal of an infinitely large number or is equal to zero. Numbers which are not infinitely large are called finite.

<u>Notation</u>. The set of all finite numbers will be denoted by M^* and the set of all infinitesimals by I^*.

<u>Theorem 21</u>. (i) A number a is infinitely large if and only if $|a| > n^*$ for all $n^* \geq 1^*$.

(ii) A number a is an infinitesimal if and only if $|a| < 1^*/n^*$ for all $n^* \geq 1^*$.

(iii) A number a is finite if and only if there exists a number $n^* \in N^*$ such that $|a| < n^*$.

Proof: (i) Let a be infinitely large. Then by Definition 14, there exists an element A of a such that A is unbounded on every element of U. Hence for every positive integer n, $\{\lambda | \lambda \in \Lambda, |A(\lambda)| \leq n\} \notin U$. Since U is an ultrafilter, $\{\lambda | \lambda \in \Lambda, |A(\lambda)| > n\} \in U$ for every $n \geq 1$. Hence $|a| > n^*$ for all $n^* \geq 1^*$. Since every step in the above proof is reversible, the converse implication also holds, proving (i) in toto. (ii) and (iii) are obvious from (i).

Corollary. $R^* \subset M^*$.

Proof: For any $r \in R^*$, there exists an element n^* of N^* such that $|r| < n^*$. This implies that $r \in M^*$.

Theorem 22. M^* is a subring of S^*.

Proof: Let a and b be any two elements of M^*. Then a and b are finite. By Theorem 21, there exist numbers n_1^* and n_2^* belonging to N^* such that $|a| < n_1^*$ and $|b| < n_2^*$. Hence $|a-b| \leq |a| + |b| < n_1^* + n_2^* = (n_1+n_2)^*$, implying that $a-b \in M^*$. Similarly, $ab \in M^*$ because $|ab| = |a||b| < n_1^* n_2^* = (n_1 n_2)^*$. This proves that M^* is a subring of S^*.

Corollary. M^* is a totally ordered integral domain.

Theorem 23. I^* is a maximal ideal in M^* and M^*/I^* is ring and order isomorphic to R^* .

Remark. By the ordering of M^*/I^* we mean, as in Definition 9, that if I^*+a and I^*+b are elements of M^*/I^* , then $I^*+a \leq I^*+b$ if and only if there exist elements x and y of I^*+a and I^*+b , respectively such that $x \leq y$. In other words, $I^*+a \leq I^*+b$ if and only if there exist elements x and y of M^* such that $x \leq y$, $x-a \in I^*$ and $y-b \in I^*$. Moreover, by Theorem 13, this is a bona fide ordering of M^*/I^* because $y \in I^*$, $x \in M^*$ and $0^* < x < y$ obviously imply that $x \in I^*$.

Proof: Let $a,b \in I^*$. Then $|a| < r$ and $|b| < r$ if $r \in R^*$ and $r > 0^*$. Hence, $|a-b| \leq |a| + |b| < 2r$ if $r \in R^*$ and $r > 0^*$. This implies that $a-b \in I^*$. Let $x \in I^*$ and $y \in M^*$. Let n^* be an element of N^* such that $|y| < n^*$. (n^* exists by Theorem 21 (iii).) Then $|yx| < rn^*$ for all positive elements r of R^* . This implies that $yx \in I^*$. Hence I^* is an ideal of M^* .

Let a be an element of M^* . Then, by Theorem 21 (iii), the set $\{r|r \in R^*, a \leq r\}$ is non-empty and has a lower bound in R^*. Let $b = \inf_{R^*} \{r|r \in R^*, a \leq r\}$. Then $b \in R^*$. Now, let r be an element of R^* . Then $r \geq a$ implies $r \geq b$ by the definition of b . Hence

$$r < b \quad \text{implies} \quad r < a \quad . \tag{23.1}$$

If $r \leq a$, then r is a lower bound of $\{r_1|r_1 \in R^*, a \leq r_1\}$,

hence $r \leq b$. Hence

$$r > b \quad \text{implies} \quad r > a \ . \tag{23.2}$$

Now let r be positive. Then $b-r < b$ and $b+r > b$. By (23.1) and (23.2), $b-r < a$ and $b+r > a$. Hence $-r < b-a < r$. Hence $b-a \in I^*$. We claim that for a given $a \in M^*$ there is exactly one element b of R^* such that $b-a \in I^*$. To prove this claim, let b_1 and b_2 be elements of R^* such that $b_1-a \in I^*$ and $b_2-a \in I^*$. Then $(b_1-a) - (b_2-a) \in I^*$ because I^* is an ideal of M^* . Hence $b_1-b_2 \in I^*$. It is obvious that 0^* is the only infinitesimal element of R^* . Hence $b_1-b_2 = 0^*$, and $b_1 = b_2$. This completes the proof of the above claim. Now, let ψ be the mapping of M^* into R^* which assigns to every element a of M^* this unique element b of R^* . Hence for $a \in M^*$, $\psi(a) \in R^*$ and $\psi(a) - a \in I^*$.

We shall now show that ψ is a ring and order homomorphism of M^* onto R^* . Let a_1 and a_2 be elements of M^* . Then $\psi(a_1) \in R^*$, $\psi(a_1) - a_1 \in I^*$, $\psi(a_2) \in R^*$, and $\psi(a_2) - a_2 \in I^*$, whence $(\psi(a_1) + \psi(a_2)) \in R^*$ and $(\psi(a_1) + \psi(a_2)) - (a_1+a_2) = (\psi(a_1) - a_1) + (\psi(a_2) - a_2) \in I^*$, since I^* is an ideal. This implies that $\psi(a_1+a_2) = \psi(a_1) + \psi(a_2)$. Likewise $\psi(a_1)\psi(a_2) \in R^*$, and $\psi(a_1)\psi(a_2) - a_1a_2 = \psi(a_1)(\psi(a_2) - a_2) + (\psi(a_1) - a_1)a_2 \in I^*$ implies that $\psi(a_1a_2) = \psi(a_1)\psi(a_2)$. If $a_1 \leq a_2$ then clearly ,
$\psi(a_1) = \inf \{r | r \in R^*, a_1 \leq r\} \leq \inf \{r | r \in R^*, a_2 \leq r\} = \psi(a_2)$.
To see that ψ is a mapping onto R^* , let a be any element of R^* ; then $a \in M^*$, since $R^* \subset M^*$, and $a-a = 0^*$, hence $a-a \in I^*$. This implies that $\psi(a) = a$. Hence ψ is a ring and order homomorphism of

M^* onto R^* . Further $\psi(a) = 0^*$ if and only if $0^* - a \in I^*$, hence if and only if $a \in I^*$. This proves that I^* is the kernel of ψ .

Now by Theorem 3A of Herstein [2], M^*/I^* is ring isomorphic to R^* . Since R^* is a field, M^*/I^* is a field. By Theorem 8, I^* is a maximal ideal of M^* . Alternatively we could have shown that I^* is the set of all singular elements of M^* , and then from Theorem 10 the same result would have followed.

Finally we need to show that M^*/I^* is also order isomorphic to R^* . If $I^* + a \leq I^* + b$ then there exist elements x and y of M^* such that $x \leq y$, $x-a \in I^*$ and $y-b \in I^*$. This implies that $\psi(x) \leq \psi(y)$, $\psi(x) = \psi(a)$ and $\psi(y) = \psi(b)$. Whence $\psi(a) \leq \psi(b)$. This completes the proof of the theorem.

The mapping ψ , the order and ring homomorphism of M^* onto R^* defined in the course of the proof of Theorem 23, will play an important role in the proof of our final theorem. Some important properties of this mapping are repeated below:

(i) $\psi(a+b) = \psi(a) + \psi(b)$,

(ii) $\psi(ab) = \psi(a)\psi(b)$,

(iii) $a \leq b$ implies that $\psi(a) \leq \psi(b)$,

(iv) $\psi(a) = 0^*$ if and only if $a \in I^*$,

(v) if $a \in R^*$, then $\psi(a) = a$.

<u>Theorem 24</u>. The following four statements are equivalent:

(i) $I^* \neq \{0^*\}$;

(ii) $M^* \neq R^*$;

(iii) $S^* \neq R^*$;

(iv) $S^* \neq M^*$.

Proof: We shall show that (i) implies (ii), (ii) implies (iii), (iii) implies (iv), and (iv) implies (i). Let $I^* \neq \{0^*\}$. Then there exists an $a \in I^* \subset M^*$ such that $a \neq 0^*$. Hence $a \in M^*$ but $a \notin R^*$, because 0^* is the only infinitesimal of R^* . Hence $M^* \neq R^*$. Let (ii) hold. Since $R^* \subset M^*$, $R^* \neq M^*$, and $M^* \subset S^*$, clearly $R^* \neq S^*$. Hence (iii) holds. Now, let $S^* \neq R^*$. This implies that there exists an $a \in S^*$ such that $a \notin R^*$. Then a is either infinite or an infinitesimal different from 0^* . In the first case $a \notin M^*$, in the second $a^{-1} \in S^*$ and $a^{-1} \notin M^*$. Hence in any case $S^* \neq M^*$. Finally, let $S^* \neq M^*$. Then there exists an infinite number $a \in S^*$. Hence a^{-1} is infinitesimal and different from 0^* . This shows that $I^* \neq \{0^*\}$.

Theorem 25. The following three conditions are mutually equivalent:

(i) S^* is a proper extension of R^* ;

(ii) There exists a mapping A of Λ into R which is unbounded on every element of U ;

(iii) U is a free ultrafilter and there exists a countable family $\{\Lambda_n | n \in N\}$ of disjoint non-empty subsets of Λ such that $\cup\{\Lambda_n | n \in N\} = \Lambda$ and $\Lambda_n \notin U$ for all $n \in N$.

Proof: We shall prove the theorem by showing that (i) implies (ii), (ii) implies (iii), and (iii) implies (i) . Let (i) hold. Then

$S^* \neq R^*$, which is equivalent to $S^* \neq M^*$. This implies that there exists an $a \in S^*$ such that $a \notin M^*$; i.e., a is infinite. Hence by Definition 14, there exists an element A of a such that A is unbounded on every element of U .

Now, let (ii) hold. Define $\Lambda_n^* = \{\lambda | \lambda \in \Lambda,\ n \leq |A(\lambda)| < n+1\}$ for $n = 0,1,\ldots$. The set of all non-negative integers n such that Λ_n^* is non-empty cannot be finite, because then A would be bounded on Λ . Now we discard those terms of the sequence $(\Lambda_n^*)_{n=o}^{\infty}$ which are equal to the empty set and denote the remaining sequence of non-empty sets by $(\Lambda_n)_{n=1}^{\infty}$. Since A is bounded on each Λ_n , $\Lambda_n \notin U$ for $n \in N$. Hence $\Lambda'_n \in U$ for $n \in N$. It is obvious that $\cup\{\Lambda_n | n \in N\} = \Lambda$. Hence $\cap\{\Lambda'_n | n \in N\} = \emptyset$. Hence $\cap U = \emptyset$. Hence U is free.

Finally let (iii) hold. Let A be the mapping of Λ into R defined by requiring that $A(\lambda) = n$ for $\lambda \in \Lambda_n$, $n \in N$. Then $a = [A] \in S^*$. Let $n \geq 2$. Then $\Lambda_1, \Lambda_2, \ldots, \Lambda_{n-1} \notin U$, hence $\Lambda'_1, \Lambda'_2, \ldots, \Lambda'_{n-1} \in U$, $\bigcap_{i=1}^{n-1} \Lambda'_i \in U$, and so $\bigcup_{i=1}^{n-1} \Lambda_i \notin U$. Since $(\bigcup_{i=1}^{n-1} \Lambda_i) \cup (\bigcup_{i=n}^{\infty} \Lambda_i) = \Lambda$, $\bigcup_{i=n}^{\infty} \Lambda_i \in U$ by Theorem 4. For $n=1$, this result is obvious. We now observe that $a \geq n^*$ for all $n^* \in N^*$, because $\{\lambda | \lambda \in \Lambda, A(\lambda) \geq n\} = \bigcup_{i=n}^{\infty} \Lambda_i \in U$. Hence $a \notin R^*$. Hence $S^* \neq R^*$.

<u>Remark</u>. It may be noted here that there exists an ultrafilter satisfying the conditons of Theorem 25. Since Λ is infinite, let $\Lambda = \bigcup_{n=1}^{\infty} \Lambda_n$, where the Λ_n are disjoint and each $\Lambda_n \neq \emptyset$. let U_o be the set of

all subsets X of Λ such that there exists a non-void finite set P of positive integers such that $X \supset \Lambda - \bigcup_{n\in P} \Lambda_n$. Then U_o is a filter on Λ . Let U be an ultrafilter on Λ finer than U_o . Then $\Lambda_n \notin U$ for $n = 1,2,\ldots$. Hence (iii) of Theorem 25 is satisfied.

CHAPTER III

THE HAHN-BANACH EXTENSION THEOREM

In what follows, X is a vector space over the field of real numbers. Little Greek letters denote scalars (or real numbers). $\overline{M}$ denotes the subspace of X generated (or spanned) by the subset M of X.

<u>Definition 15</u>. A mapping p of X into R is called a sublinear functional on X, if it has the following properties:

(i) If $x, y \in X$, then $p(x+y) \leq p(x) + p(y)$,

(ii) if $\alpha \geq 0$, and $x \in X$, then $p(\alpha x) = \alpha p(x)$.

The following theorem (see Bachman-Narici [1]) will be used in the proof of our main theorem.

<u>Theorem 26</u>. Let p be a sublinear functional on X. Let H be a proper subspace of X, and let f be a linear functional on H such that $f(x) \leq p(x)$ for every element x of H. Let x_o be an element of $X-H$, and let $G = \overline{H \cup \{x_o\}}$. Then f can be extended to a linear functional F on G with the property that $F(x) \leq p(x)$ for every element x of G.

<u>Proof</u>: If y_1 and y_2 are arbitrary elements of H, then $f(y_1) - f(y_2) = f(y_1-y_2) \leq p(y_1-y_2) \leq p(y_1+x_o) + p(-y_2-x_o)$, whence

$-p(-y_2-x_o) - f(y_2) \leq p(y_1+x_o) - f(y_1)$. From this inequality, it is clear that $\{-p(-y_2-x_o) - f(y_2) | y_2 \in H\}$ is bounded above, and that $\{p(y_1+x_o) - f(y_1) | y_1 \in H\}$ is bounded below. Let

$$a = \sup \{-p(-y_2-x_o) - f(y_2) | y_2 \in H\} , \quad \text{and}$$

$$b = \inf \{p(y_1+x_o) - f(y_1) | y_1 \in H\} .$$

Then $a \leq b$. Let c_o be a real number such that $a \leq c_o \leq b$. Then for every element y of H ,

$$-p(-y-x_o) - f(y) \leq c_o \leq p\ (y+x_o) - f(y) \quad . \tag{26.1}$$

It follows from the definition of G that every element x of G has a representation of the form $x = y+\alpha x_o$ where $y \in H$. Because $x_o \notin H$, this representation is unique. We can, therefore, define a mapping F of G into R by requiring that $F(y+\alpha x_o) = f(y) + \alpha c_o$ for $y \in H$, $\alpha \in R$. F is clearly a linear functional on G . Putting $\alpha = 0$, we find that $F(y) = f(y)$ for all $y \in H$. This shows that F is an extension of f . To show that $F(x) \leq p(x)$ for all $x \in G$, we need to show that

$$f(y) + \alpha c_o \leq p(y+\alpha x_o) \quad \text{for} \quad y \in H, \alpha \in R . \tag{26.2}$$

(26.2) is obvious if $\alpha = 0$. For $\alpha > 0$ or $\alpha < 0$, (26.2) is respectively, equivalent to the inequality

$$f(\frac{y}{\alpha}) + c_o \leq p(\frac{y}{\alpha} + x_o) \tag{26.3}$$

or

$$-f(\frac{y}{\alpha}) - c_o \leq p(-\frac{y}{\alpha} - x_o) \quad . \qquad (26.4)$$

But (26.3) as well as (26.4) follows from (26.1), with y replaced by $\frac{y}{\alpha}$. Hence (26.2) holds if $\alpha > 0$ or $\alpha < 0$. Hence (26.2) holds in any case, completing the proof.

Following Luxemburg [3 (1)] we now give the proof of the famous Hahn-Banach extension theorem. In this proof, the sign D_λ will denote the domain of λ if λ is a linear functional on a subspace of X .

Theorem 27. (Hahn-Banach) Let H be a proper subspace of X , let p be a sublinear functional on X , and let f be a linear functional on H such that, $f(x) \leq p(x)$ for all $x \in H$. Then there exists a linear functional F on X extending f such that $F(x) \leq p(x)$ for all $x \in X$.

Proof: Let Λ be the set of all linear functionals λ on subspaces of X containing H which extend f and have the property that $\lambda(x) \leq p(x)$ for all elements x of D_λ . Then Λ is non-empty, for f itself is a member of Λ . Moreover, by Theorem 26, there exist other members of Λ . For any element x of X , let $\Lambda(x) = \{\lambda | \lambda \in \Lambda, x \in D_\lambda\}$. By Theorem 26, $\Lambda(x) \neq \emptyset$ for all $x \in X$. Let $\mathcal{F} = \{\Lambda(x) | x \in X\}$. Then $\mathcal{F}$ is a non-empty set of subsets of Λ . We claim that $\mathcal{F}$ has the finite intersection property. To establish this claim, let n be a positive integer, and let x_i $(i = 1,\ldots,n)$ be elements of X . Then it follows from Theorem 26 by induction that

there exists an element λ of Λ whose domain is $\overline{H \cup \{x_i | i = 1,\ldots,n\}}$. This implies that $x_i \in D_\lambda$ for $i = 1,\ldots,n$; hence $\lambda \in \Lambda(x_i)$ for $i = 1,\ldots,n$; hence $\lambda \in \cap\{\Lambda(x_i) | i = 1,\ldots,n\}$. Hence $\cap\{\Lambda(x_i) | i = 1,\ldots,n\} \neq \emptyset$. This shows that F has the finite intersection property.

Let U be an ultrafilter on Λ containing F . U exists by Theorem 7.

Corresponding to a given element x of X , we define a mapping A_x of Λ into R as follows:

$$A_x(\lambda) = \begin{cases} \lambda(x) & \text{if} \quad \lambda \in \Lambda(x) , \\ 1 & \text{if} \quad \lambda \in \Lambda - \Lambda(x) . \end{cases}$$

Since $\lambda(x) \leq p(x)$ for $\lambda \in \Lambda$, $x \in D_\lambda$, or, equivalently, $\lambda(x) \leq p(x)$ for $x \in X$, $\lambda \in \Lambda(x)$, the definition of A_x implies that $\{\lambda | \lambda \in \Lambda, A_x(\lambda) \leq p(x)\} \supset \Lambda(x)$ for all $x \in X$. By the definition of U and the first filter-property, $\{\lambda | \lambda \in \Lambda, A_x(\lambda) \leq p(x)\} \in U$ for all elements x of X . By Definition 13, $[A_x] \leq (p(x))^*$ for all elements x of X . Likewise we get $(-p(-x))^* \leq [A_x]$ for all elements x of X . Hence $[A_x]$ is finite for all $x \in X$. Therefore $[A_x] \in M^*$ for all $x \in X$.

It is obvious that $\Lambda(x) = \Lambda$ for all $x \in H$. Consequently, for an element x of H , $A_x(\lambda) = \lambda(x)$ for all $\lambda \in \Lambda$, hence $A_x(\lambda) = f(x)$ for all $\lambda \in \Lambda$. This shows that A_x is the constant mapping of Λ onto the element $f(x)$ of R . Hence $[A_x] = (f(x))^*$ for all $x \in H$.

To establish a further property of A_x, we first observe that

(i) $\Lambda(x) \cap \Lambda(y) \subset \Lambda(x+y)$ for all $x,y \in X$,

and

(ii) $\Lambda(x) \subset \Lambda(\alpha x)$ for $x \in X$, $\alpha \in R$.

For $\lambda \in \Lambda(x) \cap \Lambda(y)$ implies that $\lambda \in \Lambda(x)$ and $\lambda \in \Lambda(y)$, hence $x \in D_\lambda$ and $y \in D_\lambda$, hence $x+y \in D_\lambda$, hence $\lambda \in \Lambda(x+y)$. This proves (i). (ii) is obvious for a similar reason. Now, in view of (i) we have $A_x(\lambda) = \lambda(x)$, $A_y(\lambda) = \lambda(y)$ and $A_{x+y}(\lambda) = \lambda(x+y)$, for every element λ of $\Lambda(x) \cap \Lambda(y)$. By the linearity of λ, $A_{x+y}(\lambda) = A_x(\lambda) + A_y(\lambda)$ for all $\lambda \in \Lambda(x) \cap \Lambda(y)$. Hence $\{\lambda | \lambda \in \Lambda, A_{x+y}(\lambda) = A_x(\lambda) + A_y(\lambda)\} \supset \Lambda(x) \cap \Lambda(y)$. Since $\Lambda(x) \cap \Lambda(y) \in \mathcal{U}$, $\{\lambda | \lambda \in \Lambda, A_{x+y}(\lambda) = A_x(\lambda) + A_y(\lambda)\} \in \mathcal{U}$, by the first filter-property. This implies that $[A_{x+y}] = [A_x + A_y]$, whence $[A_{x+y}] = [A_x] + [A_y]$.

If $\lambda \in \Lambda(x)$ then $\lambda \in \Lambda(\alpha x)$, $A_x(\lambda) = \lambda(x)$, and $A_{\alpha x}(\lambda) = \lambda(\alpha x)$. Since λ is a linear functional, $A_{\alpha x}(\lambda) = \alpha A_x(\lambda)$. Hence $\{\lambda | \lambda \in \Lambda, A_{\alpha x}(\lambda) = \alpha A_x(\lambda)\} \supset \Lambda(x)$, whence $[A_{\alpha x}] = \alpha^*[A_x]$.

We now denote by g the mapping $(x \to [A_x] | x \in X)$ of X into M^* and summarize the above results as:

(iii) $g(x) \leq (p(x))^*$ for all $x \in X$,

(iv) $g(x) = (f(x))^*$ for all $x \in H$,

(v) $g(x+y) = g(x) + g(y)$ for $x,y \in X$,

(vi) $g(\alpha x) = \alpha^* g(x)$ for $x \in X$, $\alpha \in R$.

Finally, we define a mapping F of X into R by requiring that $F(x) = \rho^{-1}(\psi(g(x)))$ for every element x of X.

(For the definitions of ρ and ψ see Theorem 20 and the proof of Theorem 23.)

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[rrd, dashed, "F"'] & M^* \arrow[r, "\psi"] & R^* \arrow[d, "\rho^{-1}"'] \\
& & R
\end{tikzcd}$$

Since $g(x) \leq (p(x))^*$ for all $x \in X$, $\psi(g(x)) \leq \psi((p(x))^*) = (p(x))^*$. Hence $F(x) = \rho^{-1}(\psi(g(x))) \leq \rho^{-1}((p(x))^*) = p(x)$. So, $F(x) \leq p(x)$ for all $x \in X$.

For every element x of H, $F(x) = \rho^{-1}(\psi(g(x))) = \rho^{-1}(\psi((f(x))^*)) = \rho^{-1}((f(x))^*) = f(x)$.

For any elements, x and y of X,

$$\begin{aligned}
F(x+y) &= \rho^{-1}(\psi(g(x+y))) = \rho^{-1}(\psi(g(x)+g(y))) \\
&= \rho^{-1}(\psi(g(x)) + \psi(g(y))) \\
&= \rho^{-1}(\psi(g(x))) + \rho^{-1}(\psi(g(y))) \\
&= F(x) + F(y) .
\end{aligned}$$

For $x \in X$, $\alpha \in R$,

$$\begin{aligned} F(\alpha x) &= \rho^{-1}(\psi(g(\alpha x))) = \rho^{-1}(\psi(\alpha^* g(x))) \\ &= \rho^{-1}(\psi(\alpha^*) \cdot \psi(g(x))) = \rho^{-1}(\alpha^* \cdot \psi(g(x))) \\ &= \rho^{-1}(\alpha^*) \cdot \rho^{-1}(\psi(g(x))) = \alpha\rho^{-1}(\psi(g(x))) \\ &= \alpha F(x) \ . \end{aligned}$$

Hence F is a linear functional on X extending f such that $F(x) \leq p(x)$ for all $x \in X$, as required.

Remark. As said in the abstract, the interesting thing about this proof is that it avoids direct use of Zorn's lemma. Whether Theorem 27 can be proved completely independently of any form of the axiom of choice is an open question.

BIBLIOGRAPHY

1. Bachman, G. and Narici, L.

Functional Analysis, Academic Press Inc., New York and London, 1966.

2. Herstein, I.N.

Topics in Algebra, Blaisdell Publishing Company, New York, 1964.

3. Luxemburg, W.A.J.

1. Two Applications of the Method of Construction by Ultrapowers to Analysis, Bull. Amer. Math. Soc. 68 (1962), pp. 416-419.

2. Non-Standard Analysis, California Institute of Technology, Pasadena, California, 1962.

4. Robinson, A.

Non-Standard Analysis, Nederl. Akad. Wetensch. Proc. Ser. A 64 (1961) = Indag Maths 23 (1961), pp. 432-440.